Soliton "molecules": Robust clusters of light bullets

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We show how to generate robust self-sustained clusters of soliton bullets-spatiotemporal (optical or matter-wave) solitons. The clusters carry an orbital angular momentum being supported by competing nonlinearities. The "atoms" forming the "molecule" are fully three-dimensional solitons linked via a staircase-like macroscopic phase. Recent progress in generating atomic-molecular coherent mixing in Bose-Einstein condensates might open potential scenarios for the experimental generation of these soliton molecules with matter-waves.

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Solitons - non-spreading, self-sustained wave packets - are at the core of nonlinear science, thus they have been investigated and observed in a variety of settings during the last two decades [1]. Today, one of the most challenging open frontiers of the field is the elucidation of complex soliton structures or "soliton molecules" to be constructed from a number of "atoms", each being a fundamental soliton. However, multi-soliton structures found so far [2]-[10] tend to self-destroy through expansion or collapse, or at best exist as meta-stable states which break apart by small perturbations. Here we reveal, for the first time to our knowledge, a physical mechanism for generating clusters which are made of stable fully three-dimensional light bullets that propagate stably over huge distances even in the presence of random perturbations in the initial conditions. The core of our approach is the use of two-color parametric solitons supported by competing nonlinearities [11, 12], which allow both, to generate stable fully three-dimensional solitons and to reduce the soliton-soliton interactions and enhancing the clusters robustness. The clusters are thus multicolored, carry orbital angular momentum, and are linked via a staircase-like macroscopic phase distribution. We present the analysis for optical spatiotemporal solitons, but our findings are intended to stimulate further theoretical and experimental research in the case of matter-waves in Bose-Einstein condensates [13]-[16].

Spatiotemporal optical solitons, the so-called "light bullets" (LBs), are self-sustained objects localized in all spatial dimensions and in time [17]-[24] (for a recent overview see Ref.[25]). They result from the simultaneous balance of diffraction and dispersion by the medium nonlinearity, and a two-dimensional version has recently been generated in quadratic nonlinear media [26]. On one hand, spatiotemporal solitons are challenging objects for fundamental research, as examples of stable localized objects in three-dimensional nonlinear fields are rare in physics. On the other hand, spatiotemporal solitons hold promise for potential applications in future ultrafast all-optical processing devices [27]-[29], where each soliton represents a bit of information and should be employed for digital operations. Multi-channel all-optical soliton networks have been proposed based on the concept of soliton clusters [30], the structures carrying many interacting individual solitons, recently introduced for two-dimensional solitons in saturable nonlinear media [6].

Soliton clusters can be viewed as a nontrivial generalization of "spinning" solitons (or doughnut-like vortices) [31]-[37] and necklace-ring beams [2]-[5], and they also appear in the study of active nonlinear systems such as externally driven optical cavities [38],[39]. But the soliton clusters investigated so far tend to be unstable or meta-stable under the action of small perturbations. We have recently shown in the case of two-dimensional spatial solitons, that the competition between quadratic and cubic nonlinearities reduces the strength of the soliton-soliton interactions, thus making spatial soliton clusters more robust under propagation [40]. Here, we consider for the first time the case of clusters made of fully three-dimensional light bullets, and show that they propagate stably over huge distances even in the presence of random perturbations.

We consider the propagation of two-color (fundamental wave and second harmonic) LB "molecules" (see the sketch in Fig. 1) in a bulk dispersive medium with competing quadratic and cubic (Kerr) self-defocusing nonlinearities. Under suitable conditions, the interaction between a fundamental frequency (FF) signal and its second harmonic (SH), in the presence of the self-defocusing cubic nonlinearity, dispersion and diffraction in the (3+1)-

dimensional geometry, can be described by the reduced model [41]-[43]

$$
\begin{aligned}
i\frac{\partial u}{\partial Z}+ \quad & \frac{1}{2}\left(\frac{\partial^2 u}{\partial X^2}+\frac{\partial^2 u}{\partial Y^2}+\frac{\partial^2 u}{\partial T^2}\right)+u^*\,v \\
& -\alpha(|u|^2+2|v|^2)u\;=0, \\
i\frac{\partial v}{\partial Z}+ \quad & \frac{1}{4}\left(\frac{\partial^2 v}{\partial X^2}+\frac{\partial^2 v}{\partial Y^2}+\sigma\frac{\partial^2 v}{\partial T^2}\right)-\beta v+u^2 \\
& -2\alpha(2|u|^2+|v|^2)v\;=0. \qquad (1)
\end{aligned}
$$

Here, T, X, Y and Z are the normalized reduced time, transverse spatial coordinates, and propagation distance, u and v are envelopes of the FF and SH fields, α measures the strength of the defocusing cubic nonlinearity, and β is a phase mismatch between the FF and SH waves. Here * stands for the complex conjugate of a complex field. Equations (1) assume different group-velocity dispersion coefficients at the two frequencies, σ being their ratio, and assumes that the temporal group-velocity mismatch between them has been compensated. Notice that Eqs.(1) correspond to the simplest model of light propagation in media with competing nonlinearities (e.g., it assumes a non-critical, type I, oo or ee wave interaction). In practice, the strength of each of the possible cross-phase-modulations depends critically on the crystalline symmetry of the particular material employed through the polarizations of the fields involved, hence the actual value of the relevant elements of the nonlinear susceptibility tensor. However, Eqs. (1) are expected to capture the essential physics behind the soliton cluster evolution.

The interaction Hamiltonian of the system is:

$$
\begin{aligned}
H \;=\; & \frac{1}{2}\int\!\!\int\!\!\int \{(|u_X|^2+|u_Y|^2+|u_T|^2)+ \\
& \frac{1}{4}(|v_X|^2+|v_Y|^2+\sigma|v_T|^2)+\beta|v|^2-(u^{*2}v+u^2v^*) \\
& +\alpha(|u|^4+4|u|^2|v|^2+|v|^4)\}\,dXdYdT \qquad (2)
\end{aligned}
$$

is a conserved quantity during evolution. Its absolute and local minima correspond to stable and metastable configurations, respectively.

Circular light-bullet necklaces were constructed as superposition of N fundamental spatiotemporal solitons with different phases such that the overall phase jump around the core is a multiple of 2π (see Fig. 1). We thus have:

$$
\begin{aligned}
u(Z=0) \;&=\; \sum_{n=1}^{N} u_0(\vec{r}-\vec{r}_n)e^{i\phi_n}\ , \\
v(Z=0) \;&=\; \sum_{n=1}^{N} v_0(\vec{r}-\vec{r}_n)e^{2i\phi_n}\ , \qquad (3)
\end{aligned}
$$

where u_0 , v_0 are the fundamental solitons at both frequencies, $\vec{r}_n$ are the soliton locations, whereas the soliton phases at those points are $\phi_n = 2n\pi M/N$ and $2\phi_n$, respectively. Here M determines the full phase twist around the cluster and plays the role of a topological charge (”spin”). We have considered circular soliton arrays, i.e. equally spaced ”atoms” displaced on a circle of radius R_0 . First, by appropriate numerical techniques (a standard band-matrix algorithm to deal with the resulting two-point boundary-value problem) we have found the families of stationary solutions to Eqs. (1) - i.e. the fundamental (non-spinning) three-dimensional spatiotemporal solitons (u_0 ,v_0). In fact, the stationary three-dimensional parametric soliton can be well approximated by a super-Gaussian ”ansatz” with suitable chosen amplitudes and widths for both the FF and SH fields.

The parameters that play an important role in the dynamics of the LB ”molecules” are the necklace topological charge M, the number of ”pearls” N forming the cluster, the initial radius of the necklace R_0, the energy E_{LB} of each constituent soliton, the wave-vector mismatch β and the strength of the defocusing cubic nonlinearity α. In almost all of our calculations we have considered the phase-matching of the interacting waves, taking thus $\beta = 0$. We have also set $\sigma = 1$, assuming equal dispersions at both frequencies, and $\alpha = 0.2$ as the dynamical equations possess scaling properties with respect to α. By increasing the strength of the defocusing cubic nonlinearity one will slow down the interaction between the constituent ”atoms”. Taking into account that the medium with competing nonlinearities supports stable spatiotemporal vortices (vortex tori) with unit topological charge when their energy exceeds a threshold [44], we have studied in detail the dynamics of soliton ”molecules” which have the total energy exceeding the corresponding stability threshold energy of the vortex soliton. Because the energy threshold for the existence of a stable vortex torus at $\alpha = 0.2$ is $E_{th} \approx 9120$, we have considered here clusters with $N = 5$ and $N = 6$ solitons, each constituent having the energy $E_{LB} = 2100$, whereas for the cluster with $N = 4$ ”atoms”, the individual energy $E_{LB} = 2824$ was correspondingly higher.

Firstly, we have studied the dependence of the cluster interaction Hamiltonian (or equivalently, the effective potential, defined as $H(R_0)/H(\infty)$) on the initial radius R_0 and on the necklace charge M. This quantity gives important hints when looking for soliton bound states (see, e.g. Ref. [6], [45] for a detailed analysis). While the interaction Hamiltonian for the $N = 4$ clusters does not posses any minima whatever the topological charge is (see Fig. 2(a)), for $N = 5$ and $N = 6$, local minima of the Hamiltonian are present for charge $M = 1$. For $N = 5$ the minimum is at $R_0 = 13.5$, whereas for $N = 6$ the minimum is at $R_0 = 12$. In our simulations we have added normally distributed noise with zero mean and variance $\sigma_{noise} = 0.1$ to the input ”molecules”. Keeping $M = 1$, we have varied the initial cluster radius R_0 around the minimum value given by the effective potential approach and have found a range of optimal values of the input

radii that minimize the mean radius oscillations of the soliton cluster. For $N = 6$ the value $R_0 = 12$ lies in the optimal radius interval, whereas for $N = 5$, the value $R_0 = 16$ assures small oscillations of the mean radius.

In order to check the predictions given by the study of the effective potential, we have numerically solved Eqs. (1) by using a finite-difference scheme based on a Cranck-Nicholson time discretization followed by a Newton-Picard iterative technique and the Gauss-Seidel method for solving the obtained system of equations. Transparent boundary conditions allowing the radiation to escape from the computation window have been implemented. We have monitorized the evolution of the mean radius of the cluster defined as:

$$R(Z) = \frac{1}{E}\int\int\int (X^2+Y^2+T^2)^{1/2}(|u|^2+|v|^2)dXdYdT, \tag{4}$$

where $E = \int\int\int(|u|^2 + |v|^2)dXdYdT$ is the total energy. If the initial radius R_0 of the cluster is large, then the mean radius $R(0)$ at the entrance of the nonlinear medium amounts to $R(0) \approx R_0$.

The evolution of clusters with $N = 4$ ($R_0 = 12$) and $N = 5$ ($R_0 = 16$) constituents is quite robust as shown in Fig. 3. The "molecules" undergo rotation and clean up the initial noise in the first stages of propagation. Our estimations for the angular velocity ω of the soliton clusters end up with $\omega = 0.0027$ for the $N = 4$ cluster shown in Fig. 3 and $\omega = 0.0014$ for the $N = 5$ one. Thus, cluster rotations are observable after large propagation distances. Only after thousands of diffraction lengths a quasi-periodic shrinking and expansion followed by a decay into several unequal fragments is observed as seen in Fig. 4. Our soliton clusters are much more robust than the LB clusters in quadratic and cubic saturable materials that survive only a few diffraction lengths in the presence of initial random noise.

The simulations with other necklace charges ($M = 0$, $M = 2$ and $M = 3$) for clusters composed of $N = 5$ ($R_0 = 16$) and $N = 6$ ($R_0 = 12$), show that the LBs forming the $M = 0$ "molecule" fuse in 100 - 150 propagation units, whereas the soliton clusters with net charges $M = 2$ or $M = 3$ expand indefinitely. Detailed simulations performed for the $N = 6$-light bullet clusters with $M = 2$ show that, by varying the initial cluster radius, the clusters formed with overlapping solitons ($10 < R_0 < 20$) expand rapidly whereas the clusters built with well separated LBs ($R_0 > 22$) have a moderate mean radius variation for a propagation distance over 600 diffraction lengths. Notice that for a typical diffraction length of a few mm, this corresponds to several meters, orders of magnitude larger than the feasible crystal lengths. Similar results were obtained for the non phase-matching case ($\beta \neq 0$).

We have also studied the influence of the initial phase distribution on the cluster dynamics by simulating the evolution of two configurations with identical intensity distributions but different phases.

The first one, build as per Eq. 3, having a staircase-like phase, destroys finally, after thousands of diffraction lengths, by splitting into two spatiotemporal solitons (Fig. 5(a)-(d)), while the second one, having a ramp-like phase mask (see the inset of Fig. 5(e)), develops into a vortex torus (Fig. 5(e)-(h)). Thus, we arrive at the conclusion that the key factor that impede the LB "molecule" with a staircase-like macroscopic phase to excite a vortex soliton is the sequence of the phase edge-dislocations (see the inset in Fig. 5(a)) existing between the neighboring solitons which form the cluster.

In summary, we have revealed a key physical mechanism for creating truly three-dimensional light bullet clusters which survive under random perturbations of the initial conditions. We have generated such structures numerically for a nonlinear optical medium with competing quadratic and cubic nonlinearities. The experimental demonstration of the concept with light waves faces many important challenges, including the generation of single light bullet. This goal requires the elucidation of a material setting with high quadratic nonlinearity, suitable group-velocity-dispersions and low one-photon and two-photon absorption at both FF and SH wavelengths, as well as small group-velocity-dispersion, together with adequate cubic nonlinearities. This is a formidable task, thus progress is being made slowly. In this context we would like to mention that it was shown recently that the strength of the cubic nonlinearity can be tuned by means of optical rectification [46] even though at present the technique has been developed only for one-dimensional beams.

However, although we showed the concept in the case of light waves, our study is important to other fields such as the physics of hybrid atomic-molecular Bose-Einstein condensates [47]-[54]. Indeed, recent experiments demonstrated coherent mixing of atomic-molecular condensates [54] which under suitable conditions should be approximately described by coupled equations for the macroscopic wave functions similar to Eqs. (1) [47]-[52]. Taking into account that to date the experimental observations of bright solitons in condensates are restricted to quasi-one dimensional geometries [13],[14], the matter-wave analogue of our light bullet clusters would correspond to clusters of condensate drops existing without a trap.

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FIGURE CAPTIONS

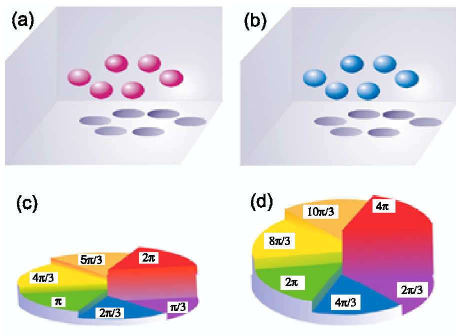
Fig. 1 Cluster composed of six spatiotemporal two-color solitons. The topological charge M of the soliton cluster is equal to one. (a) The fundamental frequency field and (b) the second harmonic field. (c) The phase distribution at fundamental frequency and (d) the phase distribution at the second harmonic.

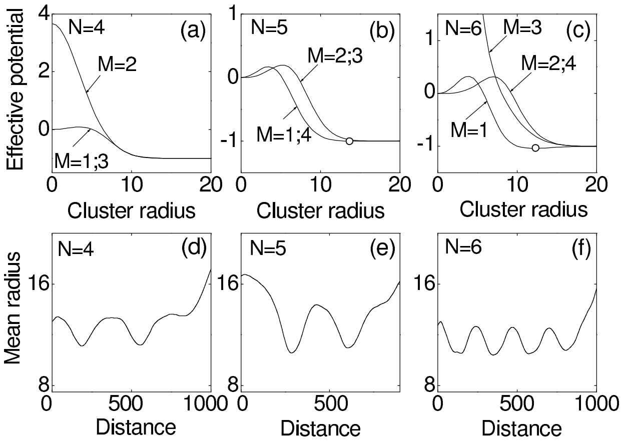
Fig. 2 Effective interaction potential versus initial cluster radius for (a) $N = 4$, (b) $N = 5$ and (c) $N = 6$ soliton clusters for different net topological charges. Typical oscillations of the mean cluster radius of solitons clusters with "spin" $M = 1$ for (d) $N = 4$, $R_0 = 12$, $E_{LB} = 2824$, (e) $N = 5$, $R_0 = 16$, $E_{LB} = 2100$ and (f) $N = 6$, $R_0 = 12$ and $E_{LB} = 2100$.

Fig. 3 Stable evolution of soliton clusters with $M = 1$ under superimposed input random noise. Shown are the contour plots for the $N = 4$ cluster: (a), $Z = 0$; (b), $Z = 25$; (c), $Z = 50$ and the contour plots for the $N = 5$ cluster: (d), $Z = 0$; (e), $Z = 25$; (f), $Z = 50$. Only the (X, Y) slices at $T = 0$ of the fundamental frequency component are shown; the second harmonic field exhibits a similar behavior. The other parameters are the same as in Fig. 2(d) for the $N = 4$ cluster and as in Fig. 2(e) for the $N = 5$ one.

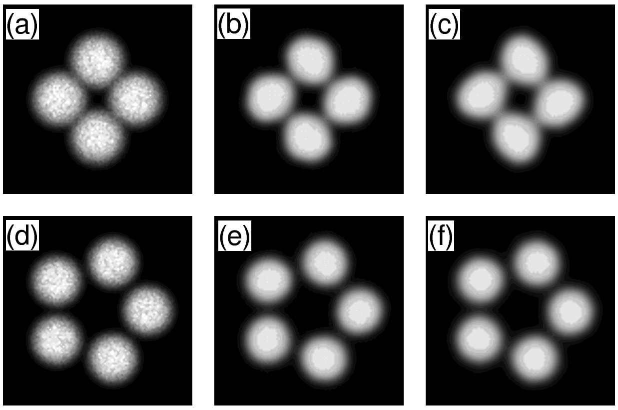
Fig. 4 Cluster evolution over long distances and the onset of symmetry breaking instability. Shown are the isosurfaces $|u| = 1.1$ for the $N = 4$ (a)-(d) and the $N = 5$ cluster (e)-(h). The parameters are the same as in Figs. 2(d) and 2(e).

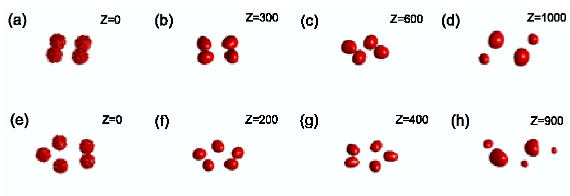
Fig. 5 Comparative evolution of two clusters with identical intensity distributions but different phase masks. The net topological charge is the same ($M = 1$) in both situations. Top panels: evolution of a six-soliton "molecule" with a step-like phase distribution; bottom panels: evolution of a six-soliton "molecule" with a ramp-like phase distribution. Shown are the isosurfaces $|u| = 1.1$. The insets in panels (a) and (e) show the initial phase mask.


(a)
(b)
(c)
5π/3
2π
4π/3
π
2π/3
π/3
(d)
10π/3
4π
8π/3
2π
4π/3
2π/3

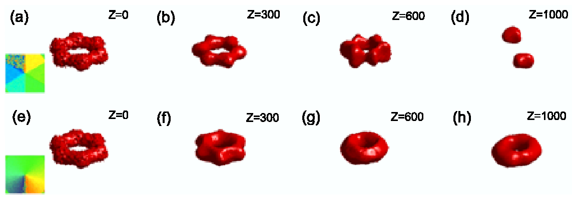


Effective potential
N=4
(a)
M=2
M=1;3
Cluster radius
N=5
(b)
M=2;3
M=1;4
Cluster radius
N=6
M=3
(c)
M=2;4
M=1
Cluster radius
Mean radius
N=4
(d)
Distance
N=5
(e)
Distance
N=6
(f)
Distance


(a)
(b)
(c)
(d)
(e)
(f)



(a)
Z=0
(b)
Z=300
(c)
Z=600
(d)
Z=1000
(e)
Z=0
(f)
Z=200
(g)
Z=400
(h)
Z=900



(a)
Z=0
(b)
Z=300
(c)
Z=600
(d)
Z=1000
(e)
Z=0
(f)
Z=300
(g)
Z=600
(h)
Z=1000